

J -kink domain walls and the DBI action

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Abstract

We study J -kink domain walls in $D = 4$ massive \mathbb{CP}^1 sigma model. The domain walls are not static but stationary, since they rotate in an internal S^1 space with a frequency ω and a momentum \mathbf{k} along the domain wall. They are characterized by a conserved current $J_\mu = (Q, \mathbf{J})$, and are classified into magnetic ($J^2 < 0$), null ($J^2 = 0$), and electric ($J^2 > 0$) types. Under a natural assumption that a low energy effective action of the domain wall is dual to the $D = 4$ DBI action for a membrane, we are lead to a coincidence between the J -kink domain wall and the membrane with constant magnetic field B and electric field \mathbf{E} . We also find that $(Q, \mathbf{J}, \omega, \mathbf{k})$ is dual to $(B, \mathbf{E}, H, \mathbf{D})$ with H and \mathbf{D} being a magnetizing field and a displacement field, respectively.

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1 Introduction

A field theory domain wall is reminiscent of a D2-brane of type IIA superstring theory. An analogy was first pointed out for domain walls in $D = 4$ supersymmetric massive hyper-Kähler sigma models [1]. In the massive $T^*\mathbb{C}P^1$ sigma model, the domain wall has collective coordinates $Z \in \mathbb{R}$ (position in a transverse direction to the domain wall) and $\phi \in S^1$ (a Nambu-Goldstone mode for a $U(1)$ global symmetry). Regarding the internal moduli ϕ as a coordinate of a “hidden” fifth dimension, a low energy effective theory for the domain wall may be thought of as an S^1 reduction of the $D = 5$ supermembrane [1]. Hence, the effective theory would be dual to an Abelian gauge theory, which is quite similar to the relation between D2-brane in ten dimensions and M2-brane in eleven dimensions [2]. Another similarity was found about the Higgs mechanism on the domain walls: It was found that a low energy effective field theory on N domain walls top of each other is $U(N)$ Yang-Mills theory [3, 4], which is again quite similar to the D-branes.

There is a further strong evidence at qualitative level: In type IIA superstring theory the superstring ending on the D2-brane is a $1/4$ Bogomol’nyi-Prasad-Sommerfield (BPS) state. This can be understood as a $1/2$ BPS BIon of the $D=10$ Dirac-Born-Infeld (DBI) action [5, 6]. A field theory counterpart of this is a $1/4$ BPS kink-lump composite solution first found in [7] and studied later in [8, 9, 10, 11, 12, 13]. This configuration can also be understood correctly as a $1/2$ BPS BIon of the $D = 4$ DBI action [7]. With this non-trivial coincidence between the field theory domain wall and the D2-brane in type IIA superstring theory, it is very plausible that the low energy effective action of the domain wall is the $D=4$ DBI action.

The purpose of this paper is clarifying further the relation between a BPS domain wall in the massive $\mathbb{C}P^1$ sigma model in four dimensions and a membrane in the $D=4$ DBI theory. Instead of studying the relation between the kink-lump and the BIon which are three dimensionally non-trivial configuration, we will focus on the flat domain wall and the flat membrane. A dyonic extension of the flat domain wall is so-called the Q -kink domain wall [1]. It is the domain wall with a conserved Noether charge Q . In Ref. [7], it was found that the Q -kink domain wall is dual to the membrane with a constant magnetic field B in the $D = 4$ DBI theory. Now, we are lead to a simple question, what is a field theory counterpart to the membrane with a constant electric field \mathbf{E} ? Having this question in mind, we will find new solutions, namely the J -kink domain walls, which possess not only the Q charge but also with a current \mathbf{J} parallel to the domain wall.

Another perspective of this paper is finding higher derivative corrections to a low energy effective theory of the domain wall. There are two ways: bottom-up and top-down approaches.

The former is a conventional method requiring a brute force, see for example [14]: First, we separate fluctuations around the domain wall background into massless and massive modes. If the massive modes are just truncated, the effective theory includes derivatives up to quadratic order, which is the so-called the moduli approximation [15]. In order to include higher order derivative corrections, one needs to expand the massive modes in terms of momenta and to integrate them out order by order. This is a straightforward task but in practice is hard to be performed. Indeed, only a first few orders have been obtained in the literature. On the other hand, the latter is just assuming the effective theory of domain wall is the $D = 4$ DBI action. As mentioned above, this is very plausible but giving a proof seems to be difficult. Therefore, we seek non-trivial checks for this. One evidence is the correspondence between the kink-lump and the BIon [7]. The results in this paper give another non-trivial evidences. Having these highly non-trivial coincidences, now we are quite sure that the DBI action is indeed the low energy effective action of the domain wall in the massive \mathbb{CP}^1 sigma model.

The paper is organized as follows. In Sec. 2, we construct the J -kink domain wall solutions in the $D = 4$ massive \mathbb{CP}^1 sigma model. Sec. 3 is devoted to finding DBI counterparts to the domain walls. In Sec. 4, we conclude the results.

2 J -kink domain walls

We are interested in topologically stable domain walls in the massive \mathbb{CP}^1 sigma model in four dimensions. The target space \mathbb{CP}^1 is isomorphic to a sphere. The Lagrangian in terms of a standard spherical coordinate $\Theta \in [0, \pi]$ and $\Phi \in [0, 2\pi)$ is given by

$$\mathcal{L} = \frac{v^2}{4} \left(-\partial_\mu \Theta \partial^\mu \Theta - \partial_\mu \Phi \partial^\mu \Phi \sin^2 \Theta - m^2 \sin^2 \Theta \right). \quad (2.1)$$

The Minkowski metric is taken to be $\eta_{\mu\nu} = (-, +, +, +)$. Mass dimensions of the parameters v and m are one. We can assume $v > 0$ and $m > 0$ without loss of generality. There are two discrete vacua at $\Theta = 0$ (the north pole) and π (the south pole). Domain walls interpolating those vacua can be obtained as solutions for the classical equations of motion

$$-\partial^\mu \partial_\mu \Theta + (m^2 + \partial_\mu \Phi \partial^\mu \Phi) \sin \Theta \cos \Theta = 0, \quad \partial^\mu (\partial_\mu \Phi \sin^2 \Theta) = 0. \quad (2.2)$$

The energy density is given by

$$\mathcal{E} = \frac{v^2}{4} \left[\dot{\Theta}^2 + (\nabla \Theta)^2 + \left(\dot{\Phi}^2 + (\nabla \Phi)^2 + m^2 \right) \sin^2 \Theta \right]. \quad (2.3)$$

The Lagrangian is invariant under a $U(1)$ global transformation $\Phi \rightarrow \Phi + \alpha$. Corresponding Noether current is given by

$$j_\mu = \frac{v^2}{2} \partial_\mu \Phi \sin^2 \Theta. \quad (2.4)$$

In what follows, we will use the current density per unit area in the x^1 - x^2 plane

$$J_\mu = (Q, \mathbf{J}), \quad Q = \int dx^3 j_0, \quad \mathbf{J} = \int dx^3 \mathbf{j}. \quad (2.5)$$

A static domain wall solution perpendicular to the x^3 -axis is given by

$$\Phi = m\phi, \quad \Theta = 2 \arctan \left[\exp \left(\pm m(x^3 - Z) \right) \right], \quad (2.6)$$

with $\phi \in S^1$ and $Z \in \mathbb{R}$ are moduli parameters with mass dimension one. We refer the solution with the upper sign as the domain wall and that with the lower sign as the anti-domain wall. A tension of the domain wall (energy per unit area in the x^1 - x^2 plane) is proportional to a topological charge Q_T

$$T = mv^2 |Q_T|, \quad Q_T \equiv \frac{1}{2} \int dx^3 \partial_3 \Theta \sin \Theta = \pm 1. \quad (2.7)$$

In what follows, we will take the upper sign, namely the domain wall.

A dyonic extension of the static kink domain wall was found in Ref. [1], as an analogue of dyons in 3+1 dimensions [16]. It is called the Q -kink domain wall. Assuming $\Theta = \Theta(x^3)$ and $\Phi = \Phi(t)$, the Q -kink domain wall solution perpendicular to the x^3 -axis can be found via the Bogomol'nyi completion of the tension $\int dx^3 \mathcal{E}$ as

$$\begin{aligned} M_Q &= \int dx^3 \frac{v^2}{4} \left[(\partial_3 \Theta - m \cos \alpha \sin \Theta)^2 + \left(\dot{\Phi} + m \sin \alpha \right)^2 \sin^2 \Theta \right. \\ &\quad \left. + 2m \partial_3 \Theta \cos \alpha \sin \Theta - 2m \dot{\Phi} \sin \alpha \sin^2 \Theta \right] \\ &\geq \sqrt{T^2 + m^2 Q^2} \cos(\alpha + \delta), \end{aligned} \quad (2.8)$$

where α is an arbitrary constant and $\tan \delta = -mQ/T$. The inequality becomes stringent when $\alpha = -\delta$. The bound is saturated for solutions of the BPS equations

$$\partial_3 \Theta = m \cos \alpha \sin \Theta, \quad \dot{\Phi} = m \sin \alpha. \quad (2.9)$$

The BPS tension of the Q -kink domain wall is given by

$$M_Q = \sqrt{T^2 + m^2 Q^2}. \quad (2.10)$$

Writing $m \sin \alpha = \omega$ and $m \cos \alpha = \sqrt{m^2 - \omega^2}$, the solution reads

$$\Phi = -\omega t + m\phi, \quad \Theta = 2 \arctan \left[\exp \left(\sqrt{m^2 - \omega^2} (x^3 - Z) \right) \right], \quad (2.11)$$

where ϕ and Z are again constants. In terms of ω , the tension and the Noether charge are expressed as

$$M_Q = \frac{mT}{\sqrt{m^2 - \omega^2}}, \quad Q = \frac{-v^2\omega}{\sqrt{m^2 - \omega^2}}, \quad (2.12)$$

with $T = mv^2$. Note that the Q -kink domain wall solution exists only for $\omega < m$. When ω reaches at m , the Q -kink domain wall becomes infinitely broad and the tension and the charge diverge.

For later convenience, let us rederive the results above in another way. First, we make an ansatz $\Theta = \Theta(x^3)$ and $\Phi = -\omega t$. This configuration indeed solves the second equation of motion for Φ in Eq. (2.2). Thus, we are left with the unknown function $\Theta(x^3)$. Plugging these into the Lagrangian, we get a reduced potential

$$\mathcal{V}_{\text{red}} = -\mathcal{L}|_{\Theta=\Theta(x^3), \Phi=-\omega t} = \frac{v^2}{4} [\partial_3 \Theta \partial^3 \Theta + (m^2 - \omega^2) \sin^2 \Theta]. \quad (2.13)$$

Let us minimize this by performing the Bogomol'nyi completion

$$\begin{aligned} \mathcal{V}_{\text{red}} &= \frac{v^2}{4} \left[\left(\partial_3 \Theta - \sqrt{m^2 - \omega^2} \sin \Theta \right)^2 + 2\sqrt{m^2 - \omega^2} \partial_3 \Theta \sin \Theta \right] \\ &\geq \frac{v^2}{2} \sqrt{m^2 - \omega^2} \partial_3 \Theta \sin \Theta. \end{aligned} \quad (2.14)$$

The bound is saturated for

$$\partial_3 \Theta = \sqrt{m^2 - \omega^2} \sin \Theta. \quad (2.15)$$

This is identical to Eq. (2.9) and is solved by the solutions given in Eq. (2.11).

Next, we generalize the Q -kink domain wall solution. It is easy to verify that the following solves the equations of motion (2.2)

$$\Phi = -k_\mu x^\mu + m\phi, \quad \Theta = 2 \arctan \left[\exp \left(\sqrt{m^2 + k^2} (x^3 - Z) \right) \right], \quad (2.16)$$

with $k_\mu = (\omega, \mathbf{k})$, $\mathbf{k} = (k_1, k_2, 0)$, and $k^2 = k_\mu k^\mu = -\omega^2 + \mathbf{k}^2$. We will call this J -kink domain wall, mimicking the Q -kink domain wall. The parameter should satisfy a condition

$$m^2 + k^2 > 0, \quad (2.17)$$

since the J -kink domain wall becomes infinity broad when $m^2 + k^2 = 0$. The tension formula in terms of k_μ is given by

$$M_J = \frac{v^2(m^2 + \mathbf{k}^2)}{\sqrt{m^2 + k^2}}. \quad (2.18)$$

When $\mathbf{k} = 0$, the J -kink domain wall reduces to the Q -kink domain wall. The conserved current density J_μ is related to the four momentum k_μ by

$$J_\mu = \frac{-v^2}{\sqrt{m^2 + k^2}} k_\mu \quad \Leftrightarrow \quad k_\mu = \frac{-m^2}{\sqrt{T^2 - m^2 J^2}} J_\mu, \quad (2.19)$$

with $J^2 = J_\mu J^\mu = -Q^2 + \mathbf{J}^2$. Note that k_μ and J_μ satisfies the following relation

$$(m^2 + k^2) (T^2 - m^2 J^2) = m^2 T^2. \quad (2.20)$$

Since $m^2 + k^2 > 0$, J^2 should satisfy

$$m^2 J^2 < T^2 \quad \Leftrightarrow \quad m^2 \mathbf{J}^2 < T^2 + m^2 Q^2. \quad (2.21)$$

Eliminating ω from (2.18), the tension formula can be expressed as

$$M_J = \sqrt{(T^2 + v^4 \mathbf{k}^2) \left(1 + \frac{Q^2}{v^4}\right)}. \quad (2.22)$$

Further eliminating \mathbf{k} , the tension formula in terms of Q and \mathbf{J} is given by

$$M_J = \frac{T^2 + m^2 Q^2}{\sqrt{T^2 - m^2 J^2}}. \quad (2.23)$$

The J -kink domain walls are classified into three types according to the sign of J^2 . We refer to the domain walls with $J^2 < 0$ as magnetic type, to those with $J^2 = 0$ as null type, and to those with $J^2 > 0$ as electric type. A reason for the names will be explained in the next section. The static domain wall (2.6) is the null type while the Q -kink domain wall (2.11) is the magnetic type. Since J^2 is a Lorentz scalar, the domain walls of the different types are not transformed each other by any Lorentz transformations.

Note that, however, the domain walls of the magnetic type ($J^2 < 0$) can be obtained by boosting the Q -kink domain wall ($J^2 = -Q^2$). In order to see this, let us boost the Q -kink domain wall with $\tilde{\omega}$ given in Eq. (2.11) with a velocity $\mathbf{u} = (u_1, u_2, 0)$

$$\Phi = -\tilde{\omega} \frac{t + \mathbf{u} \cdot \mathbf{x}}{\sqrt{1 - \mathbf{u}^2}} + m\phi, \quad \Theta = 2 \arctan \left[\exp \left(\sqrt{m^2 - \tilde{\omega}^2} (x^3 - Z) \right) \right]. \quad (2.24)$$

Rewrite $\tilde{\omega}$ and \mathbf{u} as

$$\frac{\tilde{\omega}}{\sqrt{1 - \mathbf{u}^2}} = \omega, \quad \frac{\tilde{\omega}}{\sqrt{1 - \mathbf{u}^2}} \mathbf{u} = \mathbf{k}, \quad \Rightarrow \quad \tilde{\omega}^2 = \omega^2 - \mathbf{k}^2 = -k^2. \quad (2.25)$$

Plugging this into Eq. (2.24), one reproduces the generic solutions (2.16) with $J^2 < 0$.

Contrary to the magnetic type, the domain walls of the electric type ($J^2 > 0$) cannot be obtained by boosting the Q -kink domain wall. Let us study the solution with $k_\mu = (0, \mathbf{k})$ as a representative of the electric type

$$\Phi = -\mathbf{k} \cdot \mathbf{x} + m\phi, \quad \Theta = 2 \arctan \left[\exp \left(\sqrt{m^2 + \mathbf{k}^2} (x^3 - Z) \right) \right]. \quad (2.26)$$

This has the current

$$J_\mu = \frac{-v^2}{\sqrt{m^2 + \mathbf{k}^2}}(0, \mathbf{k}). \quad (2.27)$$

We may call this \mathbf{J} -kink domain wall. The tension is given by

$$M_{\mathbf{J}} = v^2 \sqrt{m^2 + \mathbf{k}^2} = \frac{T^2}{\sqrt{T^2 - m^2 \mathbf{J}^2}} \quad (2.28)$$

Note that since Φ is the periodic variable $\Phi \sim \Phi + 2\pi$, the \mathbf{J} -kink domain wall solution is periodic along x^1 and x^2 directions with period $x^i \sim x^i + 2\pi/k^i$. Thus, this can be seen as the domain wall which wraps the “compactified” directions x^1 and x^2 with the radii $R^i = 1/k^i$. Therefore, taking an Ansatz $\Theta = \Theta(x^3)$ and $\Phi = -\mathbf{k} \cdot \mathbf{x}$ (this solves the second equation of Eq. (2.2)), the kinetic terms $(\partial_i \Phi)^2$ for the x^1 and x^2 directions give the “Kaluza-Klein” masses. This contributes to the reduced potential (2.13) as an additional mass term as

$$\mathcal{V}_{\text{red}} = \frac{v^2}{4} \left[\partial_3 \Theta \partial^3 \Theta + (m^2 + M_{\text{KK}}^2) \sin^2 \Theta \right], \quad M_{\text{KK}}^2 = \mathbf{k}^2. \quad (2.29)$$

This is the same potential as that in Eq. (2.13) with $m^2 - \omega^2$ being replaced by $m^2 + \mathbf{k}^2$. Therefore, the Bogomol’nyi completion similar to Eq. (2.14) gives the following BPS equation

$$\partial_3 \Theta = \sqrt{m^2 + \mathbf{k}^2} \sin \Theta. \quad (2.30)$$

As expected, this is solved by Eq. (2.26). Now, the generic electric solutions can be reproduced by boosting the \mathbf{J} -kink domain wall with $\tilde{\mathbf{k}}$ given in Eq. (2.26) with a velocity $\mathbf{u} = (u_1, u_2, 0)$

$$\Phi = -\frac{\tilde{\mathbf{k}} \cdot \mathbf{x} + |\tilde{\mathbf{k}}| |\mathbf{u}| t}{\sqrt{1 - \mathbf{u}^2}} + m\phi, \quad \Theta = 2 \arctan \left[\exp \left(\sqrt{m^2 + \tilde{\mathbf{k}}^2} (x^3 - Z) \right) \right]. \quad (2.31)$$

Identify $\tilde{\mathbf{k}}$ and \mathbf{u} as

$$\frac{\tilde{\mathbf{k}}}{\sqrt{1 - \mathbf{u}^2}} = \mathbf{k}, \quad \frac{|\tilde{\mathbf{k}}| |\mathbf{u}|}{\sqrt{1 - \mathbf{u}^2}} = \omega, \quad \Rightarrow \quad \tilde{\mathbf{k}}^2 = -\omega^2 + \mathbf{k}^2 = k^2. \quad (2.32)$$

Plugging these into Eq. (2.31), we return to the J -kink domain walls of the electric type.

It is interesting that the **J**-kink domain wall seems to survive even in the massless limit $m \rightarrow 0$ where the potential term in the Lagrangian vanishes. Since the potential is absent, the non-linear sigma model becomes the massless \mathbb{CP}^1 model in which whole the points on \mathbb{CP}^1 are vacua. In the massless limit, instead of the domain walls, another topological soliton, the so-called lump string, appears. In order to describe the lump strings, let us change the variable by

$$\varphi = e^{i\Phi} \tan \frac{\Theta}{2}. \quad (2.33)$$

Then the \mathbb{CP}^1 Lagrangian becomes

$$\mathcal{L} = v^2 \frac{|\partial_\mu \varphi|^2}{(1 + |\varphi|^2)^2}. \quad (2.34)$$

We consider static lump strings perpendicular to the x^3 - x^1 plane. Namely, we assume $\varphi = \varphi(x^1, x^3)$. Introducing a complex coordinate $z = x^3 + ix^1$, $\bar{z} = x^3 - ix^1$, $\partial_z = (\partial_3 - i\partial_1)/2$, and $\partial_{\bar{z}} = (\partial_3 + i\partial_1)/2$, the lump string tension can be cast into the following form

$$\begin{aligned} M_{\text{lump}} &= 2v^2 \int dx^3 dx^1 \left[2 \frac{|\partial_{\bar{z}} \varphi|^2}{(1 + |\varphi|^2)^2} + \frac{|\partial_z \varphi|^2 - |\partial_{\bar{z}} \varphi|^2}{(1 + |\varphi|^2)^2} \right] \\ &\geq 2v^2 \int dx^3 dx^1 \frac{|\partial_z \varphi|^2 - |\partial_{\bar{z}} \varphi|^2}{(1 + |\varphi|^2)^2}. \end{aligned} \quad (2.35)$$

The bound is saturated when the Bogomol'nyi equation for the lump string is satisfied

$$\partial_{\bar{z}} \varphi = 0, \quad \Rightarrow \quad \varphi = \frac{P(z)}{Q(z)}, \quad (2.36)$$

with $P(z)$ and $Q(z)$ being polynomials in z which do not have common roots. Then, the tension of the BPS lump string is given by

$$M_{\text{lump}} = 2v^2 \int dx^3 dx^1 \partial_z \partial_{\bar{z}} \log(1 + |\varphi|^2) = 2\pi v^2 k, \quad (2.37)$$

with k being a topological charge defined by $k \equiv \max\{\deg P, \deg Q\}$. Now, let us consider a special solution [17] parametrized by two real parameters k and s

$$\varphi(k, s) = (1 + s)e^{-kz} - s. \quad (2.38)$$

This is periodic in x^1 direction with period $2\pi/k$. Let us divide the x^3 - x^1 plane into domains $D_n = \{(x^3, x^1) \mid x^3 \in (-\infty, \infty), x^1 \in [2n\pi/k, (2n+1)\pi/k]\}$ for $n \in \mathbb{Z}$. Irrespective of the value of s , the solution (2.38) gives one lump string charge at each D_n

$$M_{\text{lump}@D_n} = 2v^2 \int_{D_n} dx^3 dx^1 \partial_z \partial_{\bar{z}} \log(1 + |\varphi|^2) = 2\pi v^2. \quad (2.39)$$

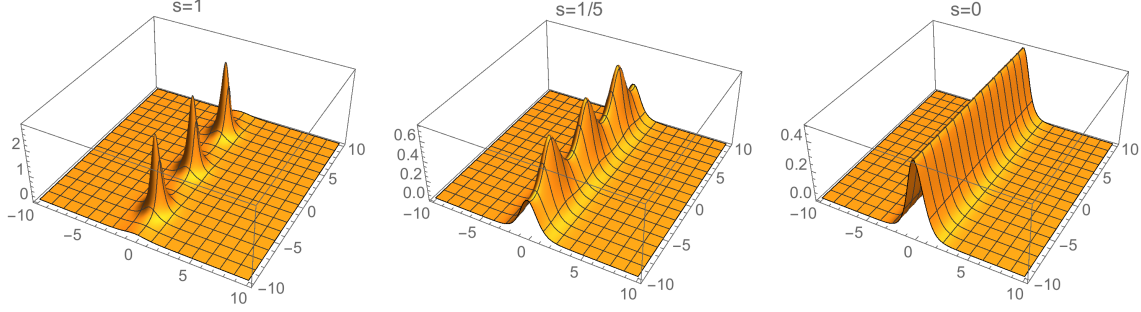


Figure 1: The energy density on the x^3 - x^1 plane for the periodic lump string configurations for φ given in Eq. (2.38) with $s = 1, 1/5, 0$.

As is shown in Fig. 1, the lumps aligned periodically on the x^1 axis for $s = 1$ merge into each other and melt into the domain wall at $s = 0$. In this way, the domain wall can appear even in the massless model as a special configuration that the lump strings are aligned periodically on a line [17]. Indeed, $\varphi(k, s = 0)$ in Eq. (2.38) is identical to the **J**-kink domain wall with $m = 0$ given in Eq. (2.26). Stability of the domain wall in the $m = 0$ limit is marginal, because it takes zero energy cost for transforming the domain wall into the lump strings.

Finally, we consider the null domain walls with $J^2 = 0$

$$\Phi = -\omega t - \mathbf{k} \cdot \mathbf{x} + m\phi, \quad \Theta = 2 \arctan \left[\exp \left(m(x^3 - Z) \right) \right], \quad (2.40)$$

with $\omega^2 = \mathbf{k}^2$. The current and the tension is given by

$$J_\mu = -\frac{v^2}{m}(\omega, \mathbf{k}), \quad M_{\text{null}} = T + \frac{m}{v^2}Q^2 = T + \frac{v^2}{m}\mathbf{k}^2. \quad (2.41)$$

This solution can be also understood from a reduced potential as done for the Q -kink domain wall in Eq. (2.13). Assuming $\Phi = -\omega t - \mathbf{k} \cdot \mathbf{x}$ and $\Theta = \Theta(x^3)$, the reduced potential reads

$$\mathcal{V}_{\text{red}} = \partial_3 \Theta \partial^3 \Theta + m^2 \sin^2 \Theta. \quad (2.42)$$

This is nothing but the sine-Gordon potential and the Bogomol'nyi completion gives us

$$\partial_3 \Theta = m \sin \Theta. \quad (2.43)$$

This is solved by Eq. (2.40). Let us take the domain wall of the null type with $k_\mu = (\xi, \xi, 0, 0)$, and boost it toward the x^1 direction. It yields the following transformation

$$\xi \rightarrow \sqrt{\frac{1-u}{1+u}} \xi. \quad (2.44)$$

Therefore, the static domain wall (2.6) is obtained in the limit of $u \rightarrow 1$.

3 The J -kink domain wall from the DBI action

In this section we will understand the domain walls with arbitrary J_μ found in the previous section from a low energy effective action of the static domain wall ($J_\mu = 0$).

For that purpose, we start with pointing out that the Q -kink domain wall solution can be understood as a boost of the static domain wall toward the hidden “fifth” direction [7]. Let us consider the massless \mathbb{CP}^1 sigma model in five dimensions,

$$S_5 = \int d^5x \tilde{v}^2 \frac{|\partial_M \varphi|^2}{(1 + |\varphi|^2)^2}, \quad (M = 0, 1, 2, 3, 4). \quad (3.1)$$

The four dimensional Lagrangian (2.1) can be derived through the Scherk-Schwarz (SS) dimensional reduction by

$$\varphi(x^\mu, w + 2\pi R_5) = e^{2\pi i m R_5} \varphi(x^\mu, w), \quad (3.2)$$

with $w = x^4$ and R_5 being the radius of the fifth direction. The mode expansion gives

$$\varphi(x^\mu, w) = e^{imw} \sum_n \varphi_n(x^\mu) e^{i \frac{n}{R_5} w}. \quad (3.3)$$

In the limit of $R_5 \rightarrow 0$, all the Kaluza-Klein tower become infinitely heavy and are decoupled, so that we are left with the lowest mode φ_0

$$\varphi(x^\mu, w) = e^{imw} \varphi_0(x^\mu), \quad \varphi_0 \equiv e^{i\Phi} \tan \frac{\Theta}{2}. \quad (3.4)$$

Plugging this into the fifth dimensional Lagrangian, we reproduce the massive \mathbb{CP}^1 sigma model as

$$S = \int d^4x \frac{v^2}{4} (-\partial_\mu \Theta \partial^\mu \Theta - \sin^2 \Theta \partial_\mu \Phi \partial^\mu \Phi - m^2 \sin^2 \Theta), \quad v^2 \equiv 2\pi R_5 \tilde{v}^2. \quad (3.5)$$

Now, the $U(1)$ isometries of the fifth direction and the target space are linked via the SS dimensional reduction. This implies that the moduli parameter ϕ appearing in Φ of the domain wall solution (2.6) should be regarded as the domain wall position $w = \phi$ in the hidden fifth direction.

Let us now “boost” the static domain wall solution (2.6) toward the fifth direction. It is done by replacing the “fifth” coordinate W by $\phi \rightarrow \frac{\phi - ut}{\sqrt{1 - u^2}}$. This yields a time dependence in to the domain wall solution

$$\Phi = m \frac{\phi - ut}{\sqrt{1 - u^2}}, \quad \Theta = 2 \arctan [\exp (\pm m(x^3 - Z))]. \quad (3.6)$$

Rewriting the boosted mass $m/\sqrt{1-u^2}$ as m and identifying $u = \omega/m$, we reproduce the Q -kink domain wall solution (2.11). Note that, since $u = \omega/m$ is a velocity, it is natural that the Q -kink domain wall solution only exist for $\omega/m \leq 1$. Furthermore, the tension of the Q -kink given in Eq. (2.12) can be written as $M_Q = T/\sqrt{1-u^2}$. This is indeed the Lorentz boosted mass formula.

In this way, it is quite natural to regard ϕ in Eq. (2.6) to be a position of the domain wall in the hidden fifth direction. Hence, a low energy effective theory of the static domain wall in the thin wall limit should be the following Nambu-Goto type Lagrangian [7],

$$\mathcal{L}_{\text{eff}} = -T \sqrt{-\det(\gamma_{\alpha\beta} + \partial_\alpha \phi \partial_\beta \phi)}, \quad (\alpha = 0, 1, 2), \quad (3.7)$$

where $\gamma_{\alpha\beta}$ is the induced metric of the domain wall given by

$$\gamma_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta}, \quad (3.8)$$

with σ^α being a world-volume coordinate and X^μ being position of the domain wall in the four dimensions.

Since the domain wall world-volume is $2 + 1$ dimensions, the effective Lagrangian (3.7) can be dualized to the $D = 4$ DBI action for a membrane by adding a BF term as

$$\mathcal{L}_{\text{eff}} = -T \sqrt{-\det(\gamma_{\alpha\beta} + \partial_\alpha \phi \partial_\beta \phi)} + \frac{\kappa T}{2} \epsilon_{\alpha\beta\gamma} \partial^\alpha \phi F^{\beta\gamma}, \quad (3.9)$$

with an Abelian field strength $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ and κ is a parameter with mass dimension -2 . We now eliminate $\partial_\alpha \phi$ from this Lagrangian by using an on-shell condition

$$\sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\beta \phi = \kappa \sqrt{1 + (\partial\phi)^2} F^\alpha, \quad F^\alpha = \frac{1}{2} \epsilon^{\alpha\beta\gamma} F_{\beta\gamma}, \quad (3.10)$$

with $(\partial\phi)^2 = \gamma^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi$. Plugging this back into the Lagrangian, we find that \mathcal{L}_{eff} can be cast into the following form

$$\mathcal{L}_{\text{eff}} = -T \sqrt{-\det(\gamma_{\alpha\beta} + \gamma^{-1} \kappa F_\alpha F_\beta)}. \quad (3.11)$$

By using the equation $\det(\gamma_{\alpha\beta} + \gamma^{-1} \kappa F_\alpha F_\beta) = \det(\gamma_{\alpha\beta} + \kappa F_{\alpha\beta})$, we finally reach at the $D = 4$ DBI Lagrangian for a membrane

$$\mathcal{L}_{\text{DBI}} = -T \sqrt{-\det(\gamma_{\alpha\beta} + \kappa F_{\alpha\beta})}. \quad (3.12)$$

In the physical gauge, this is expressed as

$$\mathcal{L}_{\text{DBI}} = -T \sqrt{-\det(\eta_{\alpha\beta} + \partial_\alpha Z \partial_\beta Z + \kappa F_{\alpha\beta})}. \quad (3.13)$$

Since we are interested in the flat domain walls with non-zero Q and \mathbf{J} , the field Z is irrelevant in the following argument. So, we will set $Z = 0$ in what follows. In other words, we will focus on non-linear electromagnetism described by the DBI Lagrangian

$$\mathcal{L}_{\text{DBI}} = -T \sqrt{1 + \frac{\kappa^2}{2} F_{\alpha\beta} F^{\alpha\beta}}. \quad (3.14)$$

The equations of motion for the electromagnetic fields read

$$\partial_\gamma \left(\frac{\kappa^2 F^{\gamma\delta}}{\sqrt{1 + \frac{\kappa^2}{2} F_{\alpha\beta} F^{\alpha\beta}}} \right) = 0. \quad (3.15)$$

One of the simplest configurations are constant electric and magnetic fields

$$F_{i0} = E_i, \quad F_{12} = B. \quad (3.16)$$

Note that, since the Lagrangian should be real valued, there is a constraint for the electric and magnetic fields

$$\kappa^2 \mathbf{E}^2 \leq 1 + \kappa^2 B^2. \quad (3.17)$$

Let us next write down the DBI Hamiltonian. First, the conjugate momentum (the displacement field) is given by

$$D_i = \frac{\partial \mathcal{L}_{\text{DBI}}}{\partial E_i} = \frac{T \kappa^2 E_i}{\sqrt{1 - \kappa^2 (\mathbf{E}^2 - B^2)}}. \quad (3.18)$$

Squaring the above equation, we get

$$\mathbf{D}^2 = \frac{T^2 \kappa^4 \mathbf{E}^2}{1 - \kappa^2 (\mathbf{E}^2 - B^2)} \quad \rightarrow \quad \kappa \mathbf{E} = \sqrt{\frac{1 + \kappa^2 B^2}{\kappa^2 T^2 + \mathbf{D}^2}} \mathbf{D}. \quad (3.19)$$

Thus the DBI Hamiltonian is given by

$$\mathcal{H}_{\text{DBI}} = \mathbf{D} \cdot \mathbf{E} - \mathcal{L}_{\text{DBI}} = \sqrt{\left(T^2 + \frac{\mathbf{D}^2}{\kappa^2} \right) (1 + \kappa^2 B^2)}. \quad (3.20)$$

Now we are ready to compare the constant electric and magnetic fields on the membrane in the DBI theory with the J -kink domain wall studied in Sec. 2. A natural identification is given by the on-shell condition (3.10), as follows. Since we set $Z = 0$, the induced metric in the physical gauge is $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$. Hence, from the on-shell condition with $\phi = -(\omega t + \mathbf{k} \cdot \mathbf{x})/m$, see Eq. (2.16), we find the following relation between $\{B, \mathbf{E}\}$ and $\{\omega, \mathbf{k}\}$

$$\kappa B = \frac{-\omega}{\sqrt{m^2 + k^2}}, \quad \kappa E_i = \frac{-\epsilon_{ij} k^j}{\sqrt{m^2 + k^2}}. \quad (3.21)$$

Comparing this with Eq. (2.19), we are lead to the following identification

$$\kappa B = \frac{Q}{v^2}, \quad \kappa E_i = -\frac{\epsilon_{ij} J^j}{v^2}. \quad (3.22)$$

The first relation between the membrane with the constant magnetic field B and the Q -kink domain wall was found in Ref. [7]. Eq. (3.22) is a generalization of this: Namely, the membrane with the constant magnetic and electric fields in the DBI theory corresponds to the J -kink domain wall via the identification (3.22).

As the J -kink domain walls are classified into three types according to the sign of the Lorentz scalar J^2 , the membrane in the DBI theory can be classified into three types by the sign of a Lorentz scalar $B^2 - \mathbf{E}^2$. The membranes with $B^2 - \mathbf{E}^2 > 0$ can be obtained by a Lorentz transformation of the membrane with $(B, \mathbf{E}) = (B, \mathbf{0})$. Similarly, the membranes with $B^2 - \mathbf{E}^2 < 0$ can be gotten from the membrane with $(B, \mathbf{E}) = (0, \mathbf{E})$. This is a reason why we christened the J -kink domain wall with $J^2 > 0$ ($J^2 < 0$) the magnetic (electric) type.

The identification (3.22) connects many quantities of the J -kink domain wall and the membrane with the constant electric and magnetic fields: For example, the constraint to Q and \mathbf{J} given in Eq. (2.21) are dual to the constraint to B and \mathbf{E} given in Eq. (3.17). In addition, the condition follows from Eq. (3.21)

$$(m^2 + k^2) (1 - \kappa^2(-B^2 + \mathbf{E}^2)) = m^2, \quad (3.23)$$

is dual to Eq. (2.20). The constraint Eqs. (2.21) for $\{Q, \mathbf{J}\}$ is also dual to the constraint (3.17) for $\{B, \mathbf{E}\}$.

Finally, let us verify the tension formulae of the J -kink domain wall and the membrane. By using the identity (3.23) and $T = mv^2$, the displacement field \mathbf{D} in Eq. (3.18) can be written as

$$\frac{D_i}{\kappa} = \frac{T}{\sqrt{1 - \kappa^2(-B^2 + \mathbf{E}^2)}} \frac{-\epsilon_{ij} k^j}{\sqrt{m^2 + k^2}} = v^2 (-\epsilon_{ij} k^j). \quad (3.24)$$

Plugging this into the Hamiltonian (3.20), we reach at the tension formula of the J -kink domain wall (2.23) expressed by Q and \mathbf{k} . Furthermore, combining Eqs. (3.19) and (3.20), the Hamiltonian of the membrane is written in terms of \mathbf{E} and B as

$$\mathcal{H}_{\text{DBI}} = \frac{T(1 + \kappa^2 B^2)}{\sqrt{1 - \kappa^2(-B^2 + \mathbf{E}^2)}}. \quad (3.25)$$

With the identification Eq. (3.22), this is equal to the tension formula (2.23) of the J -kink domain wall in terms of Q and \mathbf{J} .

In order to complete the identification, let us find a counterpart to the magnetizing field H (conjugate of B) by

$$\frac{H}{\kappa} = -\frac{1}{\kappa} \frac{\partial \mathcal{L}_{\text{DBI}}}{\partial B} = \frac{T\kappa B}{\sqrt{1 - \kappa^2(-B^2 + \mathbf{E}^2)}} = -v^2\omega. \quad (3.26)$$

Thus, the correspondence between the J -kink domain wall in the massive \mathbb{CP}^1 sigma model and the membrane with the constant electromagnetic field in the $D = 4$ DBI theory is summarized as

$$\frac{Q}{v^2} = \kappa B, \quad \frac{J_i}{v^2} = \kappa \epsilon_{ij} E^j, \quad v^2 k_i = \frac{\epsilon_{ij} D^j}{\kappa}, \quad v^2 \omega = -\frac{H}{\kappa}. \quad (3.27)$$

The expression becomes simpler if we choose $\kappa = 1/v^2$.

4 Concluding remarks

In this paper, we studied the J -kink domain wall in the massive \mathbb{CP}^1 sigma model in four dimensions, which is a generalization of the Q -kink domain wall [1]. The J -kink domain walls are classified into the three types: the magnetic type ($J^2 < 0$), the null type ($J^2 = 0$), and the electric type ($J^2 > 0$). The domain walls of the magnetic type can be obtained by boosting the Q -kink domain wall while those of the electric type can be gotten by boosting the \mathbf{J} -kink domain wall. The domain walls of the null type includes the static domain wall ($J_\mu = 0$). The generic domain walls of the null type reach at the static domain wall if they are boosted along the domain wall with the speed of light.

We explicitly showed that the Q -kink domain wall can be regarded as the domain wall which is boosted toward the hidden fifth direction. This fact strongly suggests that the low energy effective theory of the domain wall in the thin wall limit is dual to the $D = 4$ DBI action for the membrane [1, 7]. Assuming it is indeed the case, we found that the membranes with the constant electric and magnetic fields are counterpart to the J -kink domain walls. The dictionary is $(Q, \mathbf{J}, \omega, \mathbf{k}) \Leftrightarrow (B, \mathbf{E}, H, \mathbf{D})$. With this dictionary at hand, we found that many quantities, for example, the tension formulae of the domain wall and the membrane precisely coincide. These non-trivial coincidences together with the another coincidence between the kink-lump and the BIon [7] tell that the low energy effective theory of the domain wall in the massive \mathbb{CP}^1 sigma model is the $D = 4$ DBI action.

Another perspective of achievement of this paper is specifying higher derivative corrections to the low energy effective action in the moduli approximation (MA). The effective theory in MA can be obtained by promoting the zero modes Z and ϕ to be fields on the world-volume

of the domain wall. By a standard procedure, the effective Lagrangian can be found as

$$\mathcal{L}_{\text{eff}}^{\text{MA}} = -T - \frac{T}{2} (\partial_\alpha Z \partial^\alpha Z + \partial_\alpha \phi \partial^\alpha \phi). \quad (4.1)$$

This is nothing but the first two terms in expansion of the Lagrangian (3.7) in the physical gauge in terms of the derivative ∂_α . A solution is given by $Z = 0$ and $\phi = -(\omega t + \mathbf{k} \cdot \mathbf{x})/m$, which correctly reproduces Φ of the J -kink domain wall solution (2.16). However, since MA is valid only for a small ∂_α , only the solutions with small ω/m and $|\mathbf{k}|/m$ can be described by Eq. (4.1). In fact, the tension from Eq. (4.1) is

$$M_J^{\text{MA}} = T + \frac{T}{2m^2} (\omega^2 + \mathbf{k}^2), \quad (4.2)$$

which is consistent with the generic tension formula (2.18) only for small ω and $|\mathbf{k}|$. In order to reproduce the J -kink domain walls with bigger ω and $|\mathbf{k}|$, we should go beyond MA. Namely, we have to take into account higher derivative corrections. However, as is mentioned in the Introduction, finding the higher derivative corrections to MA is not an easy task. In this paper, in order to keep ourselves away from being involved into such a complicated work, we jumped to the DBI action which gives the correct tension formula to the all order of ω/m and $|\mathbf{k}|/m$. A similar strategy was recently applied to the dyonic non-Abelian vortex, and a low energy effective Lagrangian including higher derivative corrections to the all order was proposed [18].

There are several future directions. The most interesting point would be generalizing the results of this paper to multiple domain walls. In this work, we considered single domain wall in the \mathbb{CP}^1 sigma model, and found the correspondence to the Abelian DBI theory. As is well-known, N BPS domain walls exist in the massive \mathbb{CP}^N sigma model. When the N domain walls are top of each other, a non-Abelian symmetry would emerge and a non-Abelian extension of the DBI action might appear as a counterpart. Another direction is searching other J -solitons of known Q -solitons, like Q -lumps [19] and dyonic non-Abelian vortices [20] in higher dimensions.

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